IV. *On the Equation of* LAPLACE'S *Functions, &c.* *By* W. F. DONKIN, *M.A., F.R.S., F.R.A.S., Savilian Professor of Astronomy in the University of Oxford.*

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SECTION I.

THE equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

transformed to polar coordinates by putting

$$x=r\sin\theta\cos\varphi,\quad y=r\sin\theta\sin\varphi,\quad z=r\cos\theta,$$

becomes, as is well known,

$$\frac{d^2u}{d\theta^2}+\cot\theta\frac{du}{d\theta}+\frac{1}{(\sin\theta)^2}\frac{d^2u}{d\varphi^2}+r^2\frac{d^2u}{dr^2}+2r\frac{du}{dr}=0,$$

which may be written in the form

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\varphi}\right)^2+(\sin\theta)^2r\frac{d}{dr}\left(r\frac{d}{dr}+1\right)\right\}u=0; \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

and if it be assumed that

$$u=u_0+u_1r+u_2r^2+\ldots..,$$

then by substituting this value of u in either of the two equations last written, we find that u_n is a function of θ and φ satisfying the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\varphi}\right)^2+n(n+1)(\sin\theta)^2\right\}u_n=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

which, under a slightly different form, is commonly called the *Equation of* LAPLACE'S *Functions.*

This equation was first solved in finite terms by Mr. HARGREAVE *, but in a form very inconvenient for practical applications. A solution free from this objection was afterwards obtained by Professor BOOLE †, by a method explained in his memoir "On a General Method in Analysis," Philosophical Transactions, 1844. Lastly, in the second volume of the Journal referred to, the same mathematician gave two more solutions, one of which however is reducible, as he states, to Mr. HARGREAVE'S form; the other, though much more convenient than this, is still for most purposes probably less useful than that given in the first volume, from which it differs essentially in form, as well as in the method by which it is deduced.

* Philosophical Transactions, 1841.

† Cambridge and Dublin Mathematical Journal, vol. i. p. 10.

In the following pages I have treated the equation (2.) by a very simple method*. The result bears a general resemblance to Professor BOOLE'S first solution, and I conceive that the two forms must be capable of being identified by the assumption of a proper relation between the arbitrary functions; but I am not able at present to show this identity.

Some further investigations are added, which it is unnecessary to notice beforehand.

1. Putting k for $\frac{d}{d\phi}$ in equation (2.), we have

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+k^2+n(n+1)(\sin\theta)^2\right\}u_n=0. \qquad (3.)$$

In the case of $n=0$, this becomes

$$\left(\left(\sin\theta\frac{d}{d\theta}\right)^2+k^2\right)u_0=0,\ \text{or}\ \frac{d^2u_0}{dt^2}+k^2u_0=0$$

$\left(\text{where } t=\log\tan\frac{\theta}{2}\right)$. Hence

$$u_0=C_1\left(\tan\frac{\theta}{2}\right)^{k\sqrt{-1}}+C_2\left(\tan\frac{\theta}{2}\right)^{-k\sqrt{-1}}. \qquad (4.)$$

If, however, $k=0$, we should have

$$u_0=C_1+C_2\log\tan\frac{\theta}{2}, \qquad (5.)$$

* It may be worth while to give a preliminary illustration of this method (which contains no novelty except in detail), by applying a similar process to the well-known equation

$$\frac{d^2u}{dx^2}+\left(k^2-\frac{n(n+1)}{x^2}\right)u=0.$$

This may be written in the form

$$\left\{\left(D-\frac{n}{x}\right)\left(D+\frac{n}{x}\right)+k^2\right\}u=0,\ \left(D=\frac{d}{dx}\right);$$

now

$$\left(D+\frac{n}{x}\right)\left(D-\frac{n}{x}\right)=\left(D-\frac{n-1}{x}\right)\left(D+\frac{n-1}{x}\right);$$

hence if we put $u=\left(D-\frac{n}{x}\right)v$, and then operate on each side with $\left(D-\frac{n}{x}\right)^{-1}$, we have

$$\left\{\left(D-\frac{n-1}{x}\right)\left(D+\frac{n-1}{x}\right)+k^2\right\}v=\left(D-\frac{n}{x}\right)^{-1}0.$$

Assuming 0 as the value of the right-hand member, and then putting $v=\left(D-\frac{n-1}{x}\right)\omega$, and so on successively, it is obvious that we shall ultimately have

$$u=\left(D-\frac{n}{x}\right)\left(D-\frac{n-1}{x}\right)\dots\left(D-\frac{1}{x}\right)z,$$

$$(D^2+k^2)z=0;$$

hence, observing that $D-\frac{i}{x}=x^iDx^{-i}$, we obtain finally,

$$u=x^n\left(D\frac{1}{x}\right)^n(C_1\sin kx+C_2\cos kx),$$

which agrees with the result obtained by Professor BOOLE'S general method (Philosophical Transactions, 1844).

which is the complete solution of (2.) in the particular case in which u_n does not contain φ, and $n=0$.

But putting $\frac{d}{d\varphi}$ for k in (4.), and arbitrary functions of φ instead of C_1, C_2, we have

$$u_0=f\left(\varphi+\sqrt{-1}.\log\tan\frac{\theta}{2}\right)+F\left(\varphi-\sqrt{-1}.\log\tan\frac{\theta}{2}\right); \quad . \quad . \quad . \quad (6.)$$

instead of which we may evidently write

$$u_0=f\left(e^{\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right)+F\left(e^{-\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7.)$$

as well as other forms. Either of these is the complete value of u_n in the case of $n=0$.

2. Returning to the general case, if we put for shortness

$$\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2=\varrho_n,$$

it is evident that

$$\varrho_{-n}=\varrho_{n-1}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8.)$$

and also, since

$$\left(\sin\theta\frac{d}{d\theta}+n\cos\theta\right)\left(\sin\theta\frac{d}{d\theta}-n\cos\theta\right)=\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2-n^2,$$

if we put

$$\varpi_n=\sin\theta\frac{d}{d\theta}+n\cos\theta,$$

we have

$$\varpi_n\varpi_{-n}+n^2=\varrho_n, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9.)$$

from which, changing n into $-n$, and observing (8.), we get

$$\varpi_{-n}\varpi_n+n^2=\varrho_{n-1};$$

on the other hand, changing n into $n-1$ in (9.),

$$\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2=\varrho_{n-1}.$$

Comparing the last two results, we see that the operation ϖ_n possesses the following property, namely,

$$\varpi_{-n}\varpi_n+n^2=\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10.)$$

Now the equation to be solved (3.), art. 1, is

$$(\varpi_n\varpi_{-n}+n^2+k^2)u_n=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11.)$$

Let $u_n=\varpi_n v$, and let the operation ϖ_n^{-1} be performed on each side; assuming for the present that $\varpi_n^{-1}0$ may be put $=0$ without ultimate loss of generality, we obtain

$$(\varpi_{-n}\varpi_n+n^2+k^2)v=0;$$

and this, by virtue of (10.), becomes

$$(\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2+k^2)v=0;$$

which, compared with (11.), gives the relation $v=u_{n-1}$, and consequently, since $u_n=\varpi_n v$,

$$u_u=\varpi_n u_{n-1},$$

or

$$u_n=\left(\sin\theta\frac{d}{d\theta}+n\cos\theta\right)u_{n-1}.$$

This relation is true whatever be the value of n. But as we are now supposing n a positive integer, it is evident that we have

$$u_n = \varpi_n \varpi_{n-1} \dots \varpi_2 \varpi_1 u_0, \quad \dots \dots \quad (12.)$$

where u_0 is given by (6.) or (7.) of the last article in the general case, and by (5.) in the case in which u_n is a function of θ without ϕ.

This solution contains two arbitrary functions in the former case, and two arbitrary constants in the latter. It is therefore, *in general*, the complete solution. (See, however, arts. 8, 9, 17, &c.)

3. The operative symbol $\varpi_n \varpi_{n-1} \dots \varpi_2 \varpi_1$, written at length, is

$$\left(\sin\theta\frac{d}{d\theta}+n\cos\theta\right)\left(\sin\theta\frac{d}{d\theta}+(n-1)\cos\theta\right)\dots\left(\sin\theta\frac{d}{d\theta}+\cos\theta\right). \quad \dots \quad (13.)$$

Now it is evident that

$$\sin\theta\frac{d}{d\theta}+n\cos\theta=\frac{1}{(\sin\theta)^{n-1}}\frac{d}{d\theta}(\sin\theta)^n$$

(the subject of operation being of course omitted on both sides); and if this be put in the form

$$\frac{1}{(\sin\theta)^n}\sin\theta\frac{d}{d\theta}\sin\theta(\sin\theta)^{n-1},$$

it will be easily seen that the expression (13.) is equivalent to

$$\frac{1}{(\sin\theta)^n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n, \quad \dots \dots \quad (14.)$$

which may also be put in the less symmetrical form

$$\frac{1}{(\sin\theta)^{n-1}}\left((\sin\theta)^2\frac{d}{d\theta}\right)^n\sin\theta.$$

Either of these forms has the peculiarity, as compared with (13.), of being intelligible without supposing n a positive integer (setting aside the difficulties of general differentiation).

4. The results of the preceding articles may be summed up as follows:—

The complete solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2\right\}u_n=0 \text{ is}$$

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(C_1+C_2\log\tan\frac{\theta}{2}\right); \quad \dots \dots \quad (15.)$$

and the complete solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\phi}\right)^2+n(n+1)(\sin\theta)^2\right\}u_n=0 \text{ is}$$

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left\{f\left(e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)+F\left(e^{-\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)\right\}. \quad \dots \quad (16.)$$

It may be observed that (15.) is deducible from (16.) by taking $f(x)=C_1+\frac{1}{2}C_2\log x$, $F(x)=\frac{1}{2}C_2\log x$.

5. The expression (16.) may be compared with the following result of Professor BOOLE's first solution*:—

$$u_n = \mathrm{F}\left(\mu, \frac{\sqrt{(1-\mu^2)}}{\mu} e^{\phi\sqrt{-1}}\right),$$

where $\mathrm{F}(\mu, e^{\phi\sqrt{-1}})$

$$= \mu^n \left(\frac{d}{d\mu}\frac{1}{\mu}\right)^n \left\{(\mu+\mu^2)^n \psi\left(\frac{\mu e^{\phi\sqrt{-1}}}{1+\mu}\right) + (\mu-\mu^2)^n \chi\left(\frac{\mu e^{\phi\sqrt{-1}}}{1-\mu}\right)\right\}$$

and $\mu = \cos\theta$. I have stated above my impression that this form must be capable of transformation into identity with (16.); but it is probable that this can only be effected by means of some not obvious theorem of differentiation. Examples of such theorems will be seen below (arts. 14 and 16).

SECTION II.

6. I proceed to discuss some particular applications of the formulæ obtained above.

If the expression $(1-2r\cos\theta+r^2)^{-\frac{1}{2}}$ be developed in the form

$$p_0 + p_1 r + p_2 r^2 + \ldots,$$

then p_n satisfies the equation of LAPLACE's functions, and does not contain ϕ. Hence p_n must be given by the expression (15.), art. 4, if the constants be properly assumed; and since it is evident that C_2 must be $=0$ in this case, we have

$$p_n = c_n(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n . 1,$$

where c_n is a function of n.

This form of p_n may be independently verified, and c_n determined at the same time, as follows:—

Put $r(\sin\theta)^{-1} = \varrho$, then

$$(1-2r\cos\theta+r^2)^{-\frac{1}{2}} = \frac{1}{\sin\theta}\left(1+(\cot\theta-\varrho)^2\right)^{-\frac{1}{2}},$$

and if we put $\cot\theta = t$, this may be written

$$\sqrt{1+t^2}\, e^{-\rho\frac{d}{dt}} \frac{1}{\sqrt{1+t^2}},$$

where ϱ is to be treated as constant till after all operations. Now in general, if P, Q be any symbols whatever, quantitative or operative, we know that $\mathrm{P}f(\mathrm{Q})\frac{1}{\mathrm{P}} = f\left(\mathrm{PQ}\frac{1}{\mathrm{P}}\right)$. Hence it is evident that since $\sqrt{1+t^2} = \frac{1}{\sin\theta}$, and $-\frac{d}{dt} = (\sin\theta)^2\frac{d}{d\theta}$, the above expression is equivalent to

$$e^{\rho\sin\theta\frac{d}{d\theta}\sin\theta} . 1,$$

and therefore this is a symbolical form of $(1-2r\cos\theta+r^2)^{-\frac{1}{2}}$. Restoring the value of ϱ after development, we find that the coefficient of r^n is

* Cambridge and Dublin Journal, vol. i. p. 18.

$$\frac{1}{1.2.3\ldots n}(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n.1.$$

In other words, if we put $v_n=(\sin\theta)^{-n}(\sin\theta\frac{d}{d\theta}\sin\theta)^n.1$, then

$$(1-2r\cos\theta+r^2)^{-\frac{1}{2}}=v_0+v_1\frac{r}{1}+v_2\frac{r^2}{1.2}+\ldots+v_n\frac{r^n}{1.2\ldots n}+\ldots\ldots$$

(From this it is easy to deduce the relation $v_n=(2n-1)\cos\theta.v_{n-1}-(n-1)^2v_{n-2}$ by means of the known relation between the coefficients in the above series and those in the development of $(1-r^2)(1-2r\cos\theta+r^2)^{-\frac{3}{2}}$.)

7. Adopting the notation of the last article, we have

$$\begin{aligned} e^{\rho\sin\theta\frac{d}{d\theta}\sin\theta}(\sin\theta)^i &= \frac{1}{\sin\theta}e^{-\rho\frac{d}{dt}}(\sin\theta)^{i+1} \\ &= \frac{1}{\sin\theta}\left(1+(t-\rho)^2\right)^{-\frac{i+1}{2}} \\ &= (\sin\theta)^i(1-2r\cos\theta+r^2)^{-\frac{i+1}{2}}. \end{aligned}$$

Hence this theorem, which will be useful afterwards. The coefficient of $\frac{r^n}{1.2\ldots.n}$ in the development of $(1-2r\cos\theta+r^2)^{-\frac{i+1}{2}}$ is

$$(\sin\theta)^{-(n+i)}(\sin\theta\frac{d}{d\theta}\sin\theta)^n(\sin\theta)^i. \quad \ldots \ldots \ldots \quad (17.)$$

8. Let us next consider the development of the expression

$$\{1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi)+r^2\}^{-\frac{1}{2}}. \quad \ldots \ldots \quad (18.)$$

Since the coefficient of r^n is necessarily expressible in the form

$$q_0+q_1\cos\phi+q_2\cos2\phi+\ldots+q_n\cos n\phi,$$

and must also satisfy the equation (2.), art. 1, we find, on substitution in that equation, that q_i must be a solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2-i^2\right\}u_n=0. \quad \ldots \ldots \ldots \quad (19.)$$

I proceed to consider this equation in a general manner, without reference, in the first instance, to the problem immediately in hand.

Since the equation (19.) only differs from (3.), art. 1, in having $-i^2$ instead of k^2, we may apply the process used in the solution of that equation. We thus get

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(C_1\left(\tan\frac{\theta}{2}\right)^i+C_2\left(\cot\frac{\theta}{2}\right)^i\right). \quad \ldots \quad (20.)$$

Now, although this expression appears to contain two arbitrary constants, it will be shown presently, that when i is an integer and not greater than n, it really contains only *one*. This is seen immediately to be the case when $i=0$, and may be easily verified in other simple cases, such as $i=1$, $n=2$, &c.

9. In fact, the process of art. 2 gives

$$u_n = \varpi_n u_{n-1} = \varpi_n \varpi_{n-1} u_{n-2} = \ldots = \varpi_n \varpi_{n-1} \ldots \varpi_{n-m+1} \cdot u_{n-m},$$

where u_{n-m} satisfies the equation

$$\left(\varpi_{n-m}\varpi_{-(n-m)} + (n-m)^2 - i^2\right) u_{n-m} = 0;$$

and *in general* we can only arrive at an integrable form by taking $m=n$. But when i is an integer not greater than n, we shall get an integrable form by taking m so as to satisfy the condition $(n-m)^2 = i^2$. Suppose i positive and less than n, and take $m=n-i$; we then have for the solution of (19.),

$$u_n = \varpi_n \varpi_{n-1} \ldots \varpi_{i+1} \cdot u_i,$$

where u_i satisfies the equation $\varpi_i \varpi_{-i} u_i = 0$, or

$$\left(\sin\theta\frac{d}{d\theta} + i\cos\theta\right)\left(\sin\theta\frac{d}{d\theta} - i\cos\theta\right) u_i = 0:$$

the integration of this is easy, and the result is

$$u_i = C_1(\sin\theta)^i + C_2(\sin\theta)^i \int \frac{d\theta}{(\sin\theta)^{2i+1}};$$

so that, in the case considered, the integral of (19.) may be expressed in the form

$$u_n = (\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{n-i}(\sin\theta)^{2i}\left(C_1 + C_2\int d\theta(\sin\theta)^{-2i-1}\right) \quad . \quad . \quad . \quad (21.)$$

$\left(\text{for } \varpi_n \varpi_{n-1} \ldots \varpi_{i+1} \text{ is equivalent to } (\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{n-i}(\sin\theta)^i \text{ (see art. 3)}\right)$.

It is evident that the expression (21.) contains two independent arbitrary constants, whatever be the value of i. And i being supposed a positive integer, the second term will always involve a logarithmic function; hence the form (20.), which involves no such function, cannot be the complete solution of the equation (19.), and therefore we shall not really limit its generality by putting $C_2=0$. Moreover, since the coefficient of $r^n \cos i\varphi$ in the development of (18.) cannot contain any logarithmic function, it follows that, for the purpose of that development, we must put $C_2=0$ in the expression (21.).

Comparing the two forms thus obtained for the coefficient of $r^n \cos i\varphi$, we obtain incidentally the following theorem, namely,

$$\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^i \left(\tan\frac{\theta}{2}\right)^i = c(\sin\theta)^{2i}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22.)$$

c being a constant, of which the value will appear afterwards.

The preceding investigation shows that the assumption $\varpi_n^{-1}0=0$ (instead of the general value $C(\sin\theta)^{-n}$) is liable in certain cases to limit the generality of the result. I shall return to this point presently, but proceed now to complete the development of the expression (18.).

10. Since the coefficient of $r^n \cos i\varphi$ must contain θ and θ' symmetrically, and satisfy both the equation (19.) and a similar equation in θ', it is evident that it must be of the form $f(\theta)f(\theta')$. Hence, by means of the conclusions of art. 9, we arrive at the following

result: put, for convenience, $\sin\theta\frac{d}{d\theta}\sin\theta=\Theta$, $\sin\theta'\frac{d}{d\theta'}\sin\theta'=\Theta'$; then the coefficient of r^n in the development of $\left(1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi)+r^2\right)^{-\frac{1}{2}}$ may be expressed in either of the two forms,

$$\left.\begin{array}{l}\frac{1}{(\sin\theta\sin\theta')^n}\Theta^n\Theta'^n\Big\{a_0+a_1\tan\frac{\theta}{2}\tan\frac{\theta'}{2}\cos\phi+a_2\left(\tan\frac{\theta}{2}\tan\frac{\theta'}{2}\right)^2\cos 2\phi \\ \qquad+\ldots+a_n\left(\tan\frac{\theta}{2}\tan\frac{\theta'}{2}\right)^n\cos n\phi\Big\}\end{array}\right\} \quad . \quad . \quad . \quad . \quad (23.)$$

$$\left.\begin{array}{l}\frac{1}{(\sin\theta\sin\theta')^n}\{c_0\Theta^n\Theta'^n1+c_1\Theta^{n-1}\Theta'^{n-1}(\sin\theta\sin\theta')^2\cos\phi \\ \qquad+\ldots.+c_i\Theta^{n-i}\Theta'^{n-i}(\sin\theta\sin\theta')^{2i}\cos i\phi+\ldots+c_n(\sin\theta\sin\theta')^{2n}\cos n\phi\}\end{array}\right\} \quad . \quad (24.)$$

and it only remains to determine the constants.

11. For this purpose we may adopt a process of which the first part is taken from LAPLACE. Let $\cos\theta=\mu$, $\cos\theta'=\mu'$, and suppose the coefficient of $r^n\cos i\phi$ in the development of $\left(1-2r(\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}.\cos\phi)+r^2\right)^{-\frac{1}{2}}$ to be itself developed in a series of powers and products of μ and ω'. It is only necessary to ascertain the term independent of μ, μ', and the term containing the product $\mu\mu'$; and for this end we may write the above expression thus, $(1-2r(\mu\mu'+\cos\phi)+r^2)^{-\frac{1}{2}}$, since the terms μ^2, μ'^2 cannot affect the required result; and this again may (for the present purpose) be considered equal to $(1-2r\cos\phi+r^2)^{-\frac{1}{2}}+\mu\mu'r(1-2r\cos\phi+r^2)^{-\frac{3}{2}}$; and if we put $2\cos\phi=x+\frac{1}{x}$, so that $1-2r\cos\phi+r^2=(1-rx)\left(1+\frac{r}{x}\right)$, and develope in the usual manner, we easily find the following results:—

1st, if $n-i$ be even, the term independent of μ, μ' in the coefficient of $r^n\cos i\phi$ is

$$2.\frac{1^2.3^2.5^2\ldots(n+i-1)^2.1^2.3^2.5^2\ldots(n-i-1)^2}{1.2.3\ldots(n+i).1.2.3\ldots(n-i)},$$

and the term containing $\mu\mu'$ vanishes.

2nd, if $n-i$ be odd, the term independent of μ, μ' vanishes, and the term containing $\mu\mu'$ is

$$2.\frac{1^2.3^2\ldots(n+i)^2.1^2.3^2\ldots.(n-i)^2}{1.2.3\ldots(n+i).1.2.3\ldots(n-i)}\mu\mu',$$

the factor 2 to be omitted in each case when $i=0$.

12. On the other hand, the coefficient of $r^n\cos i\phi$ is (art. 10)

$$c_i(\sin\theta\sin\theta')^{-i}\Theta^{n-i}\Theta'^{n-i}(\sin\theta)^{2i}(\sin\theta')^{2i}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25.)$$

and we may find the term independent of μ, μ', and the term involving $\mu\mu'$ in the development of this, as follows:—

It is easily found, from the theorem of art. 7, that

$(\sin\theta)^{-i}\Theta^{n-i}(\sin\theta)^{2i}=1.2\ldots(n-i)(\sin\theta)^n\times$(coefficient of r^{n-i} in the development of $(1-2r\cos\theta+r^2)^{-(i+\frac{1}{2})}$); and for our present purpose we may put $\sin\theta=1$, and

$$(1-2r\cos\theta+r^2)^{-(i+\frac{1}{2})}=(1+r^2)^{-(i+\frac{1}{2})}+(2i+1)\mu r(1+r^2)^{-(i+\frac{3}{2})}.$$

Hence it is easily found that when $n-i$ is even, the term independent of μ, μ' in (25.) is

$$c_i.1^2.3^2.5^2\ldots(n-i-1)^2.(2i+1)^2(2i+3)^2\ldots(n+i-1)^2;$$

and when $n-i$ is odd, the term involving $\mu\mu'$ is

$$c_i.\mu\mu'.1^2.3^2.5^2\ldots(n-i)^2.(2i+1)^2(2i+3)^2\ldots(n+i)^2.$$

If these expressions be equated to those obtained in art. 11, we find that whether $n-i$ be even or odd the value of c_i is

$$2.\frac{1^2.3^2.5^2\ldots(2i-1)^2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)},$$

the factor 2 being omitted when $i=0$.

13. We have then the following result:—

The coefficient of $r^n\cos i\phi$ in the development of

$$\left(1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi)+r^2\right)^{-\frac{1}{2}} \text{ is}$$

$$2.\frac{1^2.3^2.5^2\ldots.(2i-1)^2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)}(\sin\theta\sin\theta')^{-n}\Theta^{n-i}\Theta'^{n-i}(\sin\theta)^{2i}(\sin\theta')^{2i}, \quad . \quad . \quad (26.)$$

where $\Theta=\sin\theta\frac{d}{d\theta}\sin\theta$, $\Theta'=\sin\theta'\frac{d}{d\theta'}\sin\theta'$; and with respect to the numerical coefficient, it is to be observed that when $i=0$ the factor 2 is to be omitted, and the numerator considered to become unity. The extreme values of i are of course 0 and n, and the term $1.2.3\ldots(n-i)$ is to be taken $=1$ when $i=n$.

14. It is now easy to ascertain the values of the coefficients a_0, a_1, &c. in the form (23.), art. 10. But first it is necessary to prove that

$$\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\tan\frac{\theta}{2}\right)^n=1.3.5\ldots(2n-1)(\sin\theta)^{2n}. \quad . \quad . \quad . \quad . \quad (27.)$$

We have proved this already (art. 9, equation (22.)), except in so far as the numerical coefficient was left undetermined, so that we have only to establish its value. Now the expression on the left of the above equation may be written

$$(\sin\theta)^{-1}\left((\sin\theta)^2\frac{d}{d\theta}\right)^n\sin\theta\left(\tan\frac{\theta}{2}\right)^n;$$

and if we put $\cot\theta=t$, so that $\tan\frac{\theta}{2}=\sqrt{1+t^2}-t$, and $(\sin\theta)^2\frac{d}{d\theta}=-\frac{d}{dt}$, the equation (27.) becomes

$$(-)^n\left(\frac{d}{dt}\right)^n\frac{(\sqrt{1+t^2}-t)^n}{\sqrt{1+t^2}}=\frac{1.3.5\ldots(2n-1)}{(1+t^2)^{\frac{2n+1}{2}}};$$

and since it is only the value of the coefficient that is in question, it will be enough to prove this when $t=0$, *i. e.* to show that the coefficient of t^n in the development of

$$\frac{(t-\sqrt{1+t^2})^n}{\sqrt{1+t^2}}$$

is $\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}$. Now, expanding the numerator by the binomial theorem, the above expression becomes

$$\frac{t^n}{\sqrt{1+t^2}}-nt^{n-1}+\frac{n(n-1)}{1.2}t^{n-2}\sqrt{1+t^2}-\frac{n(n-1)(n-2)}{1.2.3}t^{n-3}(1+t^2)+\ldots+(-)^n(1+t^2)^{\frac{n-1}{2}};$$

and if we develope each term and collect the coefficient of t^n, we find that it is

$$1+\frac{n(n-1)}{2^2}+\frac{n(n-1)(n-2)(n-3)}{2^2.4^2}+\ldots+\frac{n(n-1)\ldots(n-2i+1)}{2^2.4^2.6^2\ldots(2i)^2}+\ldots,$$

a series of which the sum* is $\frac{1.3.5\ldots(2n-1)}{1.2.3\ldots n}$. The truth of (27.) is therefore established.

15. It follows that

$$\Theta^n\left(\tan\frac{\theta}{2}\right)^i=\Theta^{n-i}\Theta^i\left(\tan\frac{\theta}{2}\right)^i=1.3\ldots(2i-1)\Theta^{n-i}(\sin\theta)^{2i},$$

and consequently the coefficient of $r^n\cos i\varphi$ may also be expressed in the form

$$\frac{2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)}(\sin\theta\sin\theta')^{-n}\Theta^n\Theta'^n\left(\tan\frac{\theta}{2}\right)^i\left(\tan\frac{\theta'}{2}\right)^i, \quad . \quad . \quad (28.)$$

the factor 2 being omitted as before when $i=0$.

The law of the numerical coefficients in this case is remarkable. For if the expression $(1+\cos\varphi)^n$ be developed in the form $A_0+A_1\cos\varphi+A_2\cos 2\varphi+\ldots+A_n\cos n\varphi$, it will be found that

$$A_i=1.2.3\ldots n.1.3.5\ldots(2n-1).\frac{2}{1.2.3\ldots(n-i).1.2.3\ldots(n+i)} \text{ (omitting the 2 when } i=0),$$

so that we may express the law of the development of

$$\left(1-2r(\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\varphi)+r^2\right)^-$$

as follows:—the coefficient of r^n is

$$\frac{1}{1.2.3\ldots n.1.3.5\ldots(2n-1)}(\sin\theta)^{-n}(\sin\theta')^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\sin\theta'\frac{d}{d\theta'}\sin\theta'\right)^n U_n,$$

where

$$U_n=A_0+A_1\cos\varphi\tan\frac{\theta}{2}\tan\frac{\theta'}{2}+A_2\cos 2\varphi\left(\tan\frac{\theta}{2}\right)^2\left(\tan\frac{\theta'}{2}\right)^2+\ldots+A_n\cos n\varphi\left(\tan\frac{\theta}{2}\right)^n\left(\tan\frac{\theta'}{2}\right)^n,$$

and A_i is the coefficient of $\cos i\varphi$ in the development of $(1+\cos\varphi)^n$.

Although this form of the development follows a more simple and remarkable law than that of art. 13, it is evident that for actual calculation it would be much less convenient, since the number of operations is much greater, and the differentiations more complex. But as the complete explicit form of the development is known independently, this consideration is not of much practical importance.

* $\frac{1.3\ldots(2n-1)}{1.2\ldots n}$ is the coefficient of x^n in the development of $(1-2x)^{-\frac{1}{2}}$; now

$$(1-2x)^{-\frac{1}{2}}=\left((1-x)^2-x^2\right)^{-\frac{1}{2}}=(1-x)^{-1}+\frac{1}{2}(1-x)^{-3}x^2+\frac{1.3}{2.4}(1-x)^{-5}x^4+\ldots;$$

and if each term be developed and the coefficient of x^n collected, the result is the series in the text.

The equation (27.) probably admits of some simpler demonstration, which I have failed to perceive.

16. Professor BOOLE, in the paper above referred to, obtains the following expression for the coefficient of $r^n \cos i\varphi$; namely,

$$\frac{2f(\mu)f(\mu')}{1.2\ldots.(n-i).1.2\ldots(n+i)},$$

where $$f(\mu)=(1-\mu^2)^{\frac{i}{2}}\mu^{n-i}\left(\frac{d}{d\mu}\frac{1}{\mu}\right)^n\mu^{n+i}(1+\mu)^{n-i}. \quad (29.)$$

Here the numerical coefficients are the same as in the form just given. It follows, therefore, that the expression on the right of (29.) must be equivalent to

$$(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\tan\frac{\theta}{2}\right)^i.$$

This equivalence, and that expressed by equation (27.), art. 14, are instances of theorems by no means obvious or easy to verify directly. (See art. 5.)

For actual calculation, Professor BOOLE's form would be preferable to either of mine; for though the number of operations is much greater than in the form (26.), their result may be assigned with much greater facility. But considered merely as an analytical expression, the form (26.) is the simplest of the three.

SECTION III.

17. It was shown above (art. 9), that the expression

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(C_1\left(\tan\frac{\theta}{2}\right)^i+C_2\left(\tan\frac{\theta}{2}\right)^{-i}\right)$$

fails in certain cases to represent the complete solution of the equation

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+n(n+1)(\sin\theta)^2-i^2\right\}u_n=0,$$

namely, when i is an *integer*, and not greater than n. (Similarly, the expression $u=C_1e^{ix}+C_2e^{-ix}$ fails to give the complete solution of $\left(\left(\frac{d}{dx}\right)^2-i^2\right)u=0$, when $i=0$.)

It is desirable, therefore, to investigate the solution of the equation in a more general manner. Putting, as before, $\sin\theta\frac{d}{d\theta}+n\cos\theta=\varpi_n$, the equation to be integrated is

$$(\varpi_n\varpi_{-n}+n^2-i^2)u_n=0. \quad (30.)$$

Let $u_n=\varpi_n v$, then $$\left(\varpi_n\varpi_{-n}\varpi_n+(n^2-i^2)\varpi_n\right)v=0,$$

and *any* value of v which satisfies this, will give a value of u_n satisfying (30.). Now $\varpi_{-n}\varpi_n+n^2$ is identically the same as $\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2$ (10.), art. 2; hence the equation becomes

$$\varpi_n\left(\varpi_{n-1}\varpi_{-(n-1)}+(n-1)^2-i^2\right)v=0;$$

similarly, putting $v=\varpi_{n-1}\omega$, we get

$$\varpi_n\varpi_{n-1}\left(\varpi_{n-2}\varpi_{-(n-2)}+(n-2)^2-i^2\right)\omega=0,$$

and so on, till we have, finally,

$$u_n = \varpi_n \varpi_{n-1} \varpi_{n-2} \dots \varpi_1 . z, \quad \dots \dots \dots \quad (31.)$$

and

$$\varpi_n \varpi_{n-1} \dots \varpi_1 (\varpi_0^2 - i^2) z = 0. \quad \dots \dots \dots \quad (32.)$$

The point to be observed is, that every value of z which satisfies (32.) will give a value of u_n satisfying (30.); hence, although the complete value of z will contain $n+2$ constants, it is certain that n of them will be destroyed by the n direct operations of the expression (31.). *In general*, the two constants left are those introduced by the inverse operation $(\varpi_0^2 - i^2)^{-1}$; but in the exceptional cases noticed above, one of these disappears, and one of those introduced by the other inverse operations remains instead.

18. The complete value of u_n, in its most general form, is therefore

$$u_n = \varpi_n \varpi_{n-1} \dots \varpi_1 (\varpi_0^2 - i^2)^{-1} \varpi_1^{-1} \varpi_2^{-1} \dots \varpi_n^{-1} . 0. \quad \dots \dots \quad (33.)$$

It was shown before that $\varpi_n \varpi_{n-1} \dots \varpi_1 = (\sin\theta)^{-n} \left(\sin\theta \frac{d}{d\theta} \sin\theta \right)^n$; in like manner we have $\varpi_1^{-1} \varpi_2^{-1} \dots \varpi_n^{-1} = \left(\frac{1}{\sin\theta} \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \right)^n (\sin\theta)^n$, the latter operation being the inverse of the former.

The expression $(\varpi_0^2 - i^2)^{-1}$ is equivalent to $\left(\left(\frac{d}{dt} \right)^2 - i^2 \right)^{-1}$, if $t = \log\tan\frac{\theta}{2}$; and this may be resolved into either of the two following forms; namely,

$$\frac{1}{2} \left\{ \left(\frac{d}{dt} - i \right)^{-1} - \left(\frac{d}{dt} + i \right)^{-1} \right\},$$

or

$$\left(\frac{d}{dt} \pm i \right)^{-1} \left(\frac{d}{dt} \mp i \right)^{-1}.$$

The former, omitting the useless factor $\frac{1}{2}$, gives

$$\left(\tan\frac{\theta}{2} \right)^i \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\cot\frac{\theta}{2} \right)^i - \left(\cot\frac{\theta}{2} \right)^i \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\tan\frac{\theta}{2} \right)^i;$$

the latter, taking the lower signs, gives

$$\left(\tan\frac{\theta}{2} \right)^i \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\cot\frac{\theta}{2} \right)^{2i} \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \left(\tan\frac{\theta}{2} \right)^i.$$

Either of these expressions is to be substituted in (33.), and thus the complete value of u_n is obtained by means of $n+2$ integrations, and n differentiations. *In general* the n integrations implied in the expression $\left(\frac{1}{\sin\theta} \left(\frac{d}{d\theta} \right)^{-1} \frac{1}{\sin\theta} \right)^n 0$ are useless, so that we may substitute 0 for this expression, as was done in the process first given, without ultimate loss of completeness.

19. If we suppose i an integer, and less than n, we may stop the process of art. 17 at an earlier stage, thus

$$u_n = \varpi_n \varpi_{n-1} \dots \varpi_{i+1} z,$$

and

$$\varpi_n \varpi_{n-1} \dots \varpi_{i+1} \varpi_i \varpi_{-i} z = 0,$$

so that we now have

$$u_n = \varpi_n \varpi_{n-1} \dots \varpi_{i+1} \varpi_{-i}^{-1} \varpi_i^{-1} \varpi_{i+1}^{-1} \dots \varpi_n^{-1} . 0;$$

or, reducing as before, and observing that ϖ_{-i}^{-1} is equivalent to

$$(\sin\theta)^i\left(\frac{d}{d\theta}\right)^{-1}(\sin\theta)^{-i-1},$$

we have, finally,

$$u_n=(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{n-i}(\sin\theta)^{2i}\left(\frac{d}{d\theta}\right)^{-1}(\sin\theta)^{-2i}\left(\frac{1}{\sin\theta}\left(\frac{d}{d\theta}\right)^{-1}\frac{1}{\sin\theta}\right)^{n-i+1}.0. \quad . \quad (34.)$$

If in this expression we put 0 for the result of *all but one* of the operations indicated in the last term, we reproduce the form (21.) obtained in art. 9, which appears in fact to be always complete. The form (34.), however, as before remarked, will never contain superfluous constants.

20. It is an interesting question whether the forms thus obtained on the supposition that i is an integer, are not really *general*, inasmuch as they are expressible without reference to that supposition. I believe that they are; but I doubt whether, in the existing state of analysis, it can be proved either that they are or are not (see § IV.).

21. It is easy to obtain other forms analogous to those given in the preceding articles, containing inverse in place of direct operations. But such forms are objectionable, because they necessarily give rise to superfluous constants, the discrimination of which may be a matter of difficulty.

Section IV.

22. It was observed at the beginning of this paper that the equation

$$\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}=0,$$

when transformed to polar coordinates, may be written in the form

$$\left\{\left(\sin\theta\frac{d}{d\theta}\right)^2+\left(\frac{d}{d\varphi}\right)^2+(\sin\theta)^2 r\frac{d}{dr}\left(r\frac{d}{dr}+1\right)\right\}u=0. \quad . \quad . \quad . \quad . \quad (35.)$$

I do not know whether this form has been noticed; but it is remarkable in this respect, that if the symbol $r\frac{d}{dr}$ be replaced by n, it becomes identical with the equation discussed in the first section, and of which the solution was shown to be

$$(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left\{f\left(e^{\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right)+\mathrm{F}\left(e^{-\varphi\sqrt{-1}}\tan\frac{\theta}{2}\right)\right\}. \quad . \quad . \quad . \quad . \quad (36.)$$

It is not easy to verify, *by direct substitution*, that this expression satisfies the differential equation of which it is the solution; at least I have not yet succeeded in doing so; and even if this were accomplished, it is most likely that difficulties would remain to be overcome before we could establish the legitimacy of extending this result to the case in which n is a symbol of unlimited signification. Still it seems worth while to try, at least as an experiment, the consequence of assuming that the form (36.) will give the solution of (35.) on putting $r\frac{d}{dr}$ instead of n.

Omitting the second arbitrary function to save space, we should have (observing that r must be considered to enter the arbitrary functions in an arbitrary manner)

$$u=(\sin\theta)^{-r\frac{d}{dr}}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{r\frac{d}{dr}}f\left(r,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right). \quad \ldots \quad (37.)$$

23. There is no difficulty in the interpretation of this expression. It is only necessary to recollect that $c^{x\frac{d}{dx}}\varphi(x)=\varphi(cx)$, and we have the following result:—let $\frac{r}{\sin\theta}=\varrho$, then

$$u=f\left(\varrho\sin\theta\frac{d}{d\theta}\sin\theta,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right), \quad \ldots \quad (38.)$$

where ϱ is to be put $=\frac{r}{\sin\theta}$ *after* all other operations, and the function must be supposed to be developed as if the two symbols $\varrho\sin\theta\frac{d}{d\theta}\sin\theta$, $e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}$ were commutative, the powers of the former being always prefixed to those of the latter.

It is easy to derive from the form (38.), particular expressions which we already know to be solutions of (35.). For example, supplying the other arbitrary function, it is evident that if we take $f(x, y)=\frac{1}{2}x^n y^i$, and $F(x, y)$ the same, we get

$$u=r^n(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\left(\tan\frac{\theta}{2}\right)^i\cos i\varphi,$$

which, we know, satisfies (35.).

Again, we may take $f(x, y)=e^x$, or

$$u=e^{\rho\sin\theta\frac{d}{d\theta}\sin\theta}.1,$$

which has been already shown (art. 6) to represent

$$(1-2r\cos\theta+r^2)^{-\frac{1}{2}}.$$

These verifications afford, I think, ground to believe that the form (37.) is a true solution of the equation (35.). And here I leave the subject, hoping that it may attract the attention of some mathematician more able than myself to bring it to a satisfactory conclusion.

Note on the last paragraph.

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If the expression $f\left(\varrho\sin\theta\frac{d}{d\theta}\sin\theta,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right)$ be supposed to be developed and arranged according to powers of the term $\varrho\sin\theta\frac{d}{d\theta}\sin\theta$, each term in the result will be of the form

$$r^n(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\psi_n\left(e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right),$$

ψ_n representing a function of which the form may be considered to depend in an arbitrary manner upon the value of n. Now we know that every such term as the above satisfies the differential equation, and it may be asked, what more is required to prove the correctness of the solution? I answer, that the point in question is whether

$$\Sigma . r^n(\sin\theta)^{-n}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^n\psi_n\left(e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(A.)}$$

is equivalent to the *most general interpretation* of

$$(\sin\theta)^{-r\frac{d}{dr}}\left(\sin\theta\frac{d}{d\theta}\sin\theta\right)^{r\frac{d}{dr}}f\left(r,\ e^{\phi\sqrt{-1}}\tan\frac{\theta}{2}\right);$$

and, if not, whether the most general interpretation would satisfy the Differential Equation (35.).

The expression (A.) gives, in fact, no new result, being an evident consequence of the conclusions of the former sections.

Postscript (*Added May* 29, 1857).

Mr. Cayley has been kind enough to communicate to me direct verifications of the equation (27.), art. 14, and of the identity referred to in art. 16. Assuming a formula established in Mr. Cayley's paper "On certain Formulæ for Differentiation," &c.*, the former of the two theorems just mentioned is easily obtained, the latter not without a good deal of trouble. The investigations of Section II. may be compared with some of those in the Introduction to Murphy's 'Electricity;' but I have not at present examined the latter with a view to such comparison, and therefore merely mention the subject.

* Cambridge and Dublin Journal, vol. ii. p. 124, equation (2.).